

Generalized Contraction Mapping and Fixed Point Theorems in Complete Metric Space

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ABSTRACT: We prove Unique fixed point theorems for self-maps satisfying some contractive type conditions. In this paper we prove the contraction mapping theorems states that a strict contraction on a complete Metric Space has a unique fixed point . The generalization of contraction mapping in complete metric space which include some fixed point results in Banach [5], Cric L. B. [8], Kannan [14], Dolhare U.P.[1,2,9], Edelstein [11] , Das and Gupta [10] , in complete metric space as a special cases.

KEYWORDS: Fixed points , Complete Metric Space , Contraction maps etc.

I. INTRODUCTION

The fixed point theory is an important and major topic of the nonlinear functional analysis and finds some nice applications to other branches of mathematics for proving the existence and uniqueness theorems to some nonlinear problems. The famous French Mathematician H. Poincare (1854) first recognized the importance of the study of fixed point theorems and nonlinear problems.

In 1922 Banach proved his celebrated fixed point theorem for Contraction Mappings in Complete Metric Space.

II. HISTORY OF FIXED POINT THEORY

Poincare [17] was the first to work in this field Fixed Point Theory. Then Brouwer [4] in 1912, proved fixed point theorem for the solution of the equation $f(x) = x$. He also proved fixed point theorem for a square, a sphere and their n-dimensional parts which means further extended by Kakutani [15] in 1922, Banach [5] proved that a contraction mapping in complete metric space possessed a unique fixed point.

In 1965 Presic S.B. [16] and Lateron contraction mapping principle developed by Kannan [14] in 1968. The theorems on contraction mapping developed by A Meir and Emmett Keeler [3] in 1969. Futher in 1975 Kolmogorov A.N. and S.V. Fomin [13] and Dass B.K., Gupta S. [10] extend the Banach Contraction principle through rational expression. Further Banach Contraction Mapping principle generalized by Dolhare U.P. [9] and Cric L.B., presic S.B. [8] in 2007.

III. FIXED POINT AND CONTRACTION

Definition 3.1 :Dolhare U.P[9]: Let X be a set and $f: X \rightarrow X$ be a map. A point $x \in X$ is called a fixed point of f if $f(x) = x$.

i.e. If f is defined on the real number by $f(x) = x^2 - 7x + 12$. We know that $x = 3, 4$ are roots of the equation. Let us

consider $f(x) = x$. Where $f(x) = \frac{x^2 + 12}{7}$ then $x = 3$ and $x = 4$ are two fixed points of $f(x)$.

Definition 3.2 :Dolhare U.P[9] Let (X, d) be a Metric Space. A mapping $f: X \rightarrow X$ is called a contraction or contraction mapping if there exists a number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y) \quad \dots \dots \dots (3.1)$$

Where $x, y \neq 0$ For all $x, y \in X$ and α is called the contraction coefficient.

Thus, a contraction maps points closer together in particular, for every $x \in X$, and any $r > 0$ all points y in the ball $B_r(x)$ are map into a ball $B_s(f_x)$ with $s < r$. Some times a map satisfying equation (3.1) with $\alpha = 1$ is also called a contraction, and then map satisfying (3.1) with $\alpha < 1$ is called a strict contraction. It follows from (3.1) that a contraction mapping is uniformly continuous.

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If $f: X \rightarrow X$, then a point $x \in X$ such that

$$f(x) = x \quad \dots \dots \dots (3.2)$$

is called a fixed point of f .

The contraction mapping theorem states that a strict contraction on a complete metric space has a unique fixed point. The contraction mapping theorem is only one example of fixed point theorems. There are fixed point theorems for maps satisfying (3.1) with $\alpha = 1$, the Schauder fixed point theorem states that a continuous mapping on a convex compact subset of Banach space has a fixed point.

Definition 3.3 : [A Meir and Emmett Keeler] [3] Let (X, ρ) be a complete Metric space and f be a mapping of X it to itself. If there exists an $\alpha < 1$ and for all $x, y \in X$ such that

$$\rho(f(x), f(y)) \leq \alpha \cdot \rho(x, y), \quad \alpha < 1$$

Then $f(\xi) = \xi$ then f has unique fixed point.

Dolhare U.P. generalized the following theorem for complete metric space as follows.

Theorem 3.1 : Dolhare U.P.[1] Let (X, d) be a complete metric space and $T: X \rightarrow X$ be satisfying

$$d(T_x, T_y) \leq \alpha \cdot d(x, y), \quad \forall x, y \in X \quad \dots \dots \dots (3.3)$$

Where $0 \leq \alpha < 1$. Then T has a unique fixed point in X .

Theorem 3.2 : Dolhare U.P. and Bele C.D.[2] Let f be a self maps of f -orbitally complete D - metric space X satisfying

$$(x, y, z)\rho(C, f_y, f_z) = \lambda \rho \text{ (where } 0 \leq \lambda \leq 1 \text{ then } f \text{ has a unique fixed point.}$$

Theorem 3.3 : Dolhare U.P.[9] :Let (X, ρ) be a complete metric space and f be a self map on X such that f^2 is continuous if $g: f(x) \rightarrow X$ such that $g f(x) \subset f^2(x)$ and $g(f(x)) = f(g(x))$ both sides are defined for all $x, y \in f(x)$. Then f and g have unique common fixed point.

Most important results in this direction has been obtained by S.B. Presic [16] by generalizing the Banach contraction mapping principle.

Theorem 3.4 : (S.B. Presic [16]) Let (X, d) be a complete metric space, k is a positive integer and $T: X^k \rightarrow X$ a mapping satisfying the following contractive type condition

$$d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + \dots + q_k d(x_k, x_{k+1}) \quad \dots \dots \dots (3.4)$$

For every $x_1, x_2, x_3, \dots, x_k, x_{k+1}$ where $q_1, q_2, q_3, \dots, q_k$ are nonnegative constants such that $q_1 + q_2 + q_3 + \dots + q_k < 1$ then there exists a unique point x in X such that

$$T(x, x, \dots, x) = x$$

Moreover if $x_1, x_2, x_3, \dots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$,

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$$

Then the sequence $\{x_n\}$ is convergent and

$$\lim x = T(\lim x, \lim x, \dots, \lim x)$$

Ciric and Presic [8] generalized theorem 3.3 as follows,

Theorem 3.5 : Let (X, d) be a complete metric space, k is a positive integer and $T: X^k \rightarrow X$ a mapping satisfying the following contractive type condition

$$d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \lambda \max d(x_i, x_{i+1}), \quad 1 \leq i \leq k \quad \dots \dots \dots (3.5)$$

For every $x_1, x_2, x_3, \dots, x_k, x_{k+1}$ in X where $\lambda \in (0, 1)$ is constants, then there exists a point x in X such that

$$T(x, x, \dots, x) = x$$

Moreover if $x_1, x_2, x_3, \dots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}$,

$$x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$$

Then the sequence $\{x_n\}$ is convergent and

$$\lim x = T(\lim x, \lim x, \dots, \lim x)$$

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If in addition, we suppose that on diagonal $\Delta \subset X^k$,

$$d(T(u, u, \dots, u), T(v, v, \dots, v)) < d(u, v) \quad \dots \dots \dots (3.6)$$

Holds for all $u, v \in X$, with $u \neq v$ then x is the unique point in X with $T(x, x, \dots, x) = x$.

Caccioppoli [7] extends the Banach's Principle for mapping T converge in a complete metric space X . Provided that for each $n \geq 1$ there exists a constant C_n such that

$$d(T^n(x), T^n(y)) \leq C_n d(u, v) \quad \dots \dots \dots (3.7)$$

For every $x, y \in X$ where $\sum C_n < \infty$

In 1962, Rakotch [19] generalized Banach contraction principle as follows

Let X be a complete metric space and $f: X \rightarrow X$ satisfies.

$$f(f(x), f(y)) \leq \lambda (d(x, y), d(x, y)) \quad \dots \dots \dots (3.8)$$

For all $x, y \in X$, where $\alpha: \mathbb{R}^+ \rightarrow [0,1]$ is monotonically decreasing, then f has a unique fixed point \bar{x} and $\{f^n(x)\}$ converges to \bar{x} for each $x \in X^n$.

Edelstien [11] generalized Banach contraction mapping principle in more general cases. while Sehgal [20] extended the concept of contraction mapping. Sehgal obtained the fixed point of a continuous function T of a complete metric space (X, d) into itself for which there exist positive integer $n = n(x)$ satisfying the following inequality

$$d(T^n x, T^n y) \leq \beta d(u, v), \quad \forall x, y \in X \ \& \ 0 < \beta < 1 \quad \dots \dots \dots (3.9)$$

Theorem 3.6 : (Sehgal V.M.) Let (X, d) be a complete metric space, and $f: X \rightarrow X$ a continuous mapping satisfying the condition ; there exists a $k > 1$ such that for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$

$$d(f^{n(x)}(y), f^{n(x)}(x)) \leq k d(y, x) \quad \dots \dots \dots (3.10)$$

Then f has a unique fixed point u and $f^n(x_0) \rightarrow u$ for each $x_0 \in X$.

Sehgal's result further generalized by Iseki, Sharma, Guseman [12]. Further Banach contraction principle is generalized by Rakotch [19], Boyd and wong [6] in 1968, Kannan [14] obtained fixed point for self mapping T on complete metric space X satisfying the contractive condition that for all $x, y \in X$, there exists $\lambda \in [0,1/2]$ such that

$$d(T_x, T_y) \leq \lambda [d(x, T_x) + d(y, T_y)] \quad \dots \dots \dots (3.11)$$

In 1962 Edelstein generalized Banach contraction principle by taking $\alpha = 0$.

Theorem 3.7 : (Edelstein [11]) Let (X, d) be a complete metric space and let $T: X \rightarrow X$ satisfying

$$d(T^{p+1}x, T^{p+2}y) \leq \frac{\alpha d(T^{p+1}y, T^{p+2}y)[1 + d(T^p x, T^{p+1}x)]}{1 + d(T^p x, T^{p+1}y)} + 1 + \beta d(T^p x, T^{p+1}x) \quad \dots \dots \dots (3.12)$$

$\forall x, y \in X$ and for non-negative constants α, β with $\alpha + \beta < 1$ and for any non-negative integer p . Then T has a unique fixed point.

Recently Dass and Gupta [10] (1975) generalized Banach contraction principle for mapping T satisfying

$$d(T_x, T_y) \leq \alpha d(y, T_y) \frac{[1 + d(x, T_x)]}{1 + d(x, y)} + \beta d(x, y) \quad \dots \dots \dots (3.13)$$

For all $x, y \in X$ and for $\alpha, \beta > 0, \alpha + \beta < 1$

There is a nice discussion of the contraction mapping and some applications in Kolmogorov and Fomin [13] for given function $f: X \rightarrow X$ a fixed point of f is a point $x \in X$ such that $f(x) = x$. A well known topological theorem asserting the existence of at least one fixed point is the Brouwer theorem.

Theorem 3.8 : (Brouwer [4]) If B is a closed ball in \mathbb{R}^n and $f: B \rightarrow B$ is a continuous map then f has a fixed point in B .

The conclusion of Banach theorem holds more generally from the following condition of weakly uniformly strict condition i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that

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$$\epsilon \leq \rho(x, y) < \epsilon + \delta \text{ implies } \rho(f(x), f(y)) < \epsilon$$

Theorem 3.9 : (A Meir and Emmett Keeler [3]) Let (X, ρ) be a complete metric space and f a mapping of X has a unique fixed point ξ . Moreover for any $x \in X$

$$\lim_{n \rightarrow \infty} f^n(x) = \xi$$

IV. MAIN RESULT

We generalized Banach contraction principle as follows which is generalization of theorem 3.5 using (3.11) and (3.13) is as follows.

Theorem 4.1 : Let (X, d) be a complete metric space and let $f: X \rightarrow X$ satisfying

$$d(f^{n+1}x, f^{n+2}y) \leq \frac{\lambda d(f^{n+1}y, f^{n+2}y)[1 + d(f^n x, f^{n+1}x)]}{1 + d(f^n x, f^{n+1}y)} + \gamma d(f^n x, f^{n+1}x) \dots \dots \dots (3.14)$$

$\forall x, y \in X$ and for positive constants λ, γ with $\lambda + \gamma < 1$ and for any non-negative integer n . Then f has a unique fixed point.

Proof: We prove the above theorem for $n = 0$, then inequality (3.14) becomes

$$d(fx, f^2y) \leq \frac{\lambda d(fy, f^2y)[1 + d(x, fx)]}{1 + d(x, fy)} + \gamma d(x, fx)$$

By taking sequence of the element $\{x_n\} \in X$ as follows

$$x_n = f^n x_0, \quad x_{n+1} = f x_n, \quad n = 0, 1, 2, \dots, x_n \in X$$

Then,

$$d(x_1, x_2) = d(fx_0, f^2x_0) \leq \frac{\lambda d(fx_0, f^2x_0)[1 + d(x_0, fx_0)]}{1 + d(x_0, fx_0)} + \gamma d(x_0, fx_0)$$

$$d(x_1, x_2) \leq \frac{\gamma}{1 - \lambda} d(x_0, fx_0) \dots \dots \dots (3.15)$$

Similarly,

Then,

$$d(x_2, x_3) = d(fx_1, f^2x_1) \leq \frac{\lambda d(fx_1, f^2x_1)[1 + d(x_1, fx_1)]}{1 + d(x_1, fx_1)} + \gamma d(x_1, fx_1)$$

$$d(x_1, x_2) \leq \frac{\gamma}{1 - \lambda} d(x_1, fx_1) \leq \left(\frac{\gamma}{1 - \lambda}\right)^2 d(x_0, fx_0) \dots \dots \dots (3.16)$$

From (3.15) and (3.16) in general we get,

$$d(x_n, x_{n+1}) \leq \left(\frac{\gamma}{1 - \lambda}\right)^n d(x_0, fx_0)$$

Since $\lambda + \gamma < 1$, then it follows that $\{x_n\}$ is a Cauchy sequence and from the completeness of X it follows that $\lim_{n \rightarrow \infty} x_n = a$ where $a \in X$, then f has a unique fixed point.

Let $f: X \rightarrow X$ be any contraction, then we believe that all contractions have fixed points then $f^n(x)$ should have a fixed point. Since it is a contraction, we generalized contraction theorem 3.7 for proving fixed point theorem as follows.

Theorem 4.2 : Let (X, ρ) be a complete metric space suppose f is contraction mapping and λ is a constant for f and λ^n is the constant for f^n . Then f^n is also a contraction and f^n has a fixed point.

Proof : Firstly we will show that the theorem is true for $n = 2$. Since f is a contraction consider $\lambda < 1$, then

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

We can apply f to $f(x)$ and $f(y)$ such that

$$d(f^2(x), f^2(y)) \leq \lambda d(f(x), f(y))$$

Since $d(f(x), f(y)) \leq \lambda d(x, y)$

$$d(f^2(x), f^2(y)) \leq \lambda d(f(x), f(y)) \leq \lambda^2 d(x, y)$$

Thus,

$$d(f^2(x), f^2(y)) \leq \lambda^2 d(x, y)$$

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Since $\lambda < 1, \lambda^2 < 1$ then f^n is a contraction,

Since f^n is a contraction then it is true that f^{n+1} is also contraction

$$d(f^{n+1}(x), f^{n+1}(y)) \leq \lambda^{n+1} d(f(x), f(y)) \leq \lambda^{n+1} d(x, y)$$

Thus, $d(f^{n+1}(x), f^{n+1}(y)) \leq \lambda^{n+1} d(x, y)$

Thus by induction the theorem is true for all n . If $f(x) = x$, then

$$f^2(x) = f(f(x)) = f(x) = x$$

Then $f^n(x) = x$

Hence by induction f^n is contraction and f^n has the unique fixed point.

Theorem 4.3 : Let $f: X \rightarrow X$ be a contraction of the complete metric space (X, ρ) so that $d(f(x), f(y)) \leq \lambda d(x, y)$ for some $0 \leq \lambda \leq 1$, and let x_0 be any point X . Then the sequence $\{x_n\}$ defined by $x_{n+1} = f(x_n)$ converges to unique fixed point x . Further more for any n^{th} value we have

$$d(x_n, x) \leq \frac{\lambda^n}{1 - \lambda} d(x_0, f(x_0))$$

Then f has unique fixed point.

Proof: Let $f: X \rightarrow X$ be a contraction of the complete metric space (X, ρ) . Let x_0 be any point on X i.e. $x_0 \in X$ and define x_n iterating $x_{n+1} = f(x_n)$

We shall prove that (x_n) is a Cauchy sequence as X is complete, the sequence will converge firstly we will bound $d(x_n, x_{n+1})$. Consider for every

$$d(x_i, x_{i+1}) = d(f(x_{i-1}), f(x_i)) \leq \lambda d(x_{i-1}, x_i)$$

So that, $d(x_n, x_{n+1}) \leq \lambda d(x_n, x_{n+1})$
 $\leq \lambda^2 d(x_{n-1}, x_n)$

$$\vdots$$

$$\leq \lambda^n d(x_1, x_0)$$

Suppose that $m > n$ and find a bound for $d(x_n, x_m)$, then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq (\lambda^n + \lambda^{n+1} + \dots + \lambda^{m-1}) d(x_0, x_1) \\ &\leq (\lambda^n + \lambda^{n+1} + \dots) d(x_0, x_1) \\ &= \frac{\lambda^n}{1 - \lambda} d(x_0, x_1) \end{aligned}$$

Since $\lambda < 1$, it follows that $\frac{\lambda^n}{1 - \lambda} \rightarrow 0$ as $n \rightarrow \infty$. It follows that (x_n) is a Cauchy sequence, and since X is complete,

it follows that $x_n \rightarrow x$ for some limit $x \in X$.

Let us consider x is a fixed point. Since the sequence x_0, x_1, \dots converges to x , then the sequence x_1, x_2, \dots is also converges to x , therefore

$$f(x) = f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x$$

So that, $f(x) = x$

Now we will show that the fixed point is unique.

Suppose that both x and x' are fixed points, then

$$d(x, x') = d(f(x), f(x')) \leq \lambda d(x, x')$$

As $\lambda < 1$, the only way this can be happen is, if $d(x, x') = 0$ and so $x = x'$

Which shows that x and x' are not different and hence this function f has a unique fixed point.

V. CONCLUSION

In the present paper we used contraction mapping for to find unique fixed point of self maps in complete Metric Space.

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